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DOMAIN DECOMPOSITION & WALL LAWS

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Abstract

We shall consider in this paper the problem of simulating flows over a rough surface or flows with strong gadients near walls. We compare effective boundary conditions on smooth surfaces obtained by domain decomposition and by asymptotic expansions. Some numerical tests are presented.

1. INTRODUCTION

We consider the problem of simulating flows over rough surfaces or flows with strong gadients. As many points must go into the mesh to resolve strong gradients such simulations are expensive. This problem is as old as the Euler/Boundary layer decomposition but it happens also in other circumstances :

- a badly polished flat plate or a surface with periodic ridges like the tiles of a re-entry vehicle or the effect of trees and buildings on a meteorological flow.
- Turbulent boundary layers where the viscous part of the flow dominates.

The usual answer to the two problems is given by the law of the wall :

$$u^+ = \frac{1}{\chi}\log y^+ + \beta$$

on a mean surface Σ above the physical boundary, with a different coefficient β when the surface is rough (see Cousteix(1990)). This formula is used to establish a numerically useful nonlinear Frechet boundary condition :

$$u.n = 0, \quad \frac{u.s}{\sqrt{\nu_T|\frac{\partial u}{\partial n}|}} - \frac{1}{\chi}\log(\delta\sqrt{\frac{1}{\nu_T}|\frac{\partial u}{\partial n}|}) + \beta = 0,$$

where ν_T is the turbulent viscosity.
Here we wish to show that it could be also derived from another generalized Frechet condition :

$$\sigma n_\Sigma = \nu_T(\nabla u + \nabla u^T)n - pn = c(|u|)u$$

which comes from domain decomposition and has nothing to do with wall laws.

This work is an extension of Carrau-Le Tallec(1994) Mohammadi et al (1994) and Achdou et al (1995).
The key idea is that when the solution near the rough boundary Γ is "local" and known, say $u = f(x_1, x_2, p)$ where x is the position in the domain and p is a parameter, then to obtain a boundary condition on Σ slightly above Γ (assumed tangent to $x_2 = 0$) one may differentiate f with respect to x_2, the fast variable, and eliminate p between the 2 equations.

$$\frac{\partial u}{\partial n} = -f'_{,2}(x,p) \quad \Rightarrow u = u(x, {f'_{,2}}^{-1}(\frac{\partial u}{\partial n}, x))$$

2. ROUGH WALLS

Consider the Reynolds averaged Navier-Stokes equations with a turbulence model for ν_T

$$u\nabla u + \nabla p - \nabla.(\nu_T(\nabla u + \nabla u^T)) = 0 \quad \nabla.u = 0$$

on a domain Ω^ϵ

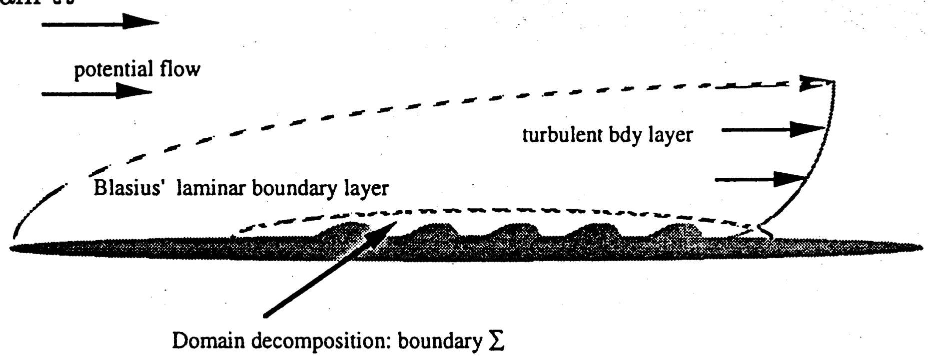

Figure 1.*Domain decomposition of the flow over a rough surface*

Following LeTallec (1991) Let us seek a solution by domain decomposition.
Let Σ be parallel to Γ, $\Omega = \Omega_o \cup \Omega_i$

Let u_i be solution in Ω_i with $u = v$ on Σ
Let u_o be solution in Ω_o with $u = v$ on Σ

We have a solution to the problem if v is such that normal stresses match :

$$\sigma_i.n = \sigma_o.n$$

Now by definition of u_i, we know that the solution is a function of v so its normal stress on the upper wall is also a function of v :

$$\sigma_i.n = F(v)$$

and the continuity of σ gives the desired boundar condition on Σ

$$\sigma_o.n = F(u_o)$$

The trouble however is that F is in general a nonlocal operator.

Periodic irregularities

For periodic irregularities F becomes approximatively local, because the solution u_i can be found by translation of the solution u' on a single cell problem with only one irregularity at the lower boundary, periodic conditions on the vertical boundaries and matching conditions $u' = v$ at the top boundary. Then this cell problem is solved for all values of v and a table is made of $\nu_T(\nabla u_i + \nabla u_i^T)n - p_i n)|_\Sigma$ versus v.

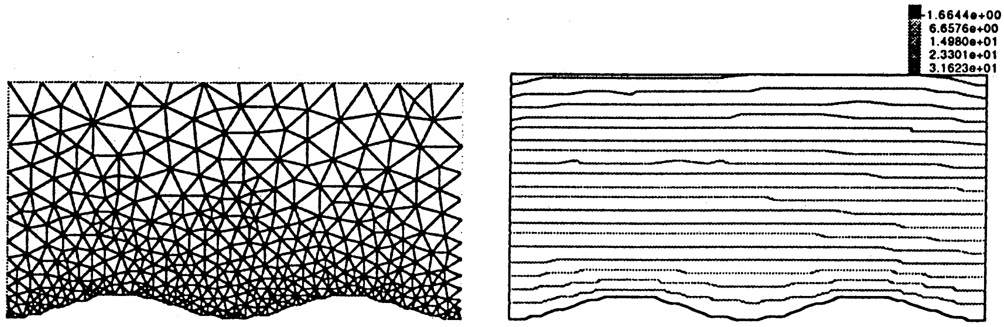

Figure 2 :*The cell proble2D Navier-Stokes eqs at Re* $= \nu^{-1} = 50$ *(Gfem(1995)).*

Remark

Notice that by the divergence theorem and Green's theorem,

$$\int_{\partial\Omega_i} [\nu_T(\nabla u + \nabla u^T)n - pn] = \int_{\Omega_i} [\nabla.(\nu_T(\nabla u + \nabla.u^T)) - \nabla p]$$
$$= \int_{\Omega_i} \nabla.(u \otimes u) = \int_{\partial\Omega_i} uu.n = 0$$

Therefore

$$-\int_{\partial\Omega\cap\bar{\Omega}_i} [\nu_T(\nabla u + \nabla u^T)n - pn] = \int_\Sigma [\nu_T(\nabla u + \nabla u^T)n - pn]$$

So $F(u)$ *is also the drag of the rough surface per unit length.* This means that tabulations of F could also be done experimentally.

3. VANISHING VISCOSITY : ANALYSIS BY ASYMPTOTIC EXPANSION

Consider the stationary Navier-Stokes equations in a domain Ω^ϵ with an oscillating boundary with period ε and viscosity $0(\varepsilon)$:

$$-\nu\varepsilon\Delta u + u.\nabla u + \nabla p = f, \quad \nabla \cdot u = 0$$

$$u|_{\partial\Omega^\varepsilon} = 0$$

In what follows the forcing term f can be replaced by a non homogeneous boundary condition . The solution is seeked by the multiple scales expansion

$$u^\varepsilon = u^0(x) + \varepsilon u^1(\frac{x}{\varepsilon}, x) + \varepsilon^2 u^2(\frac{x}{\varepsilon}, x) + ...$$

Let us assume that $\Gamma^\varepsilon = \partial\Omega^\varepsilon$ converges strongly to $\Gamma^0 = \partial\Omega^0$ and that on the oscillating part tends to a flat Γ^0 (for the influence of the radius of curvature see Achdou et al (1993). For clarity let us assume that the problem is bidimensional and that Γ^0 is on the line $x_2 = 0$.
The solution u^0 of

$$-\nu\varepsilon\Delta u + u.\nabla u + \nabla p = f, \quad \nabla \cdot u = 0$$

$$u|_{\partial\Omega^0} = 0$$

does not approximate u^ε very well because the boundary condition is not satisfied on Γ^ε.
A Taylor expansion of u^ε being

$$u^0(x_1, 0) = u^0|_{\Gamma^\varepsilon} + \varepsilon\frac{x_2}{\varepsilon}\frac{\partial u^0}{\partial n}|_{\Gamma^\varepsilon} + \varepsilon^2\frac{x_2}{\varepsilon}^2\frac{\partial^2 u^0}{\partial n^2}|_{\Gamma^\varepsilon} + ...$$

on $\Gamma^\varepsilon = \{x_1, x_2^\varepsilon(x_1)\}$ we can correct u^0 by $\chi^0\partial u_1^0/\partial n$ where χ^0 is solution of a cell problem in a semi infinite domain C of boundary $\partial C = S \cup W$

$$-\nu\Delta_y\chi^0 + \nabla_y\eta^0 = 0, \quad \nabla_y \cdot \chi^0 = 0$$

$$\chi^0|_S = y_2\vec{e^1}, \quad \chi^0 \ \ y_1 - \textit{periodic on } W, \ \ lim_{y_2\to+\infty}\chi^0 = C^0$$

where C_0 is the only constant for which χ^0 exists. Note that we do not need a corrector for $\partial u_2^\varepsilon/\partial n$ because it is zero by the divergence equation.

Remark Note that the divergence equation implies that $C_1^0 = 0$, because

$$0 = \int_{C\cap y_2<m} \nabla_y \cdot \chi^0 = \int_{\partial C\cap y_2<m} \chi^0 \cdot n = \int_{y_2=m} \chi^0 \cdot n \to C_2^0.$$

Finally notice that $u^0 - \chi_1^0\partial u^0/\partial n \approx u^0 - \chi_1^0\partial u^0/\partial n$ when $x_2 >> \varepsilon$. So if we introduce u^1 solution of

$$-\nu\varepsilon\Delta u + u.\nabla u + \nabla p = f, \quad \nabla \cdot u = 0$$

$$u.s|_{\partial\Omega^0} + \varepsilon C_1^0\frac{\partial u.s}{\partial n} = 0, \quad u.n|_{\partial\Omega^0} = 0$$

the error

$$r^\varepsilon = u^\varepsilon(x) - u^1(x) - \varepsilon(\chi^0(\frac{x}{\varepsilon}) - C^0)\frac{\partial u_1^1}{\partial n}(x_1, 0)$$

$$\rho^\varepsilon = p^\varepsilon(x) - p^1(x) - \varepsilon(\eta^0(\frac{x}{\varepsilon}) - C^0)\frac{\partial u^1}{\partial n}(x_1, 0)$$

is likely to be small. In effect when the viscosity is of order 1 we can show (see appendix) that the error is $O(\varepsilon^{3/2})$.
To compute the second order corrector we notice that on the oscillating boundary we have

$$r^\varepsilon = e^1\frac{\varepsilon^2}{2}(\frac{x_2}{\varepsilon})^2\frac{\partial^2 u_1^1}{\partial n^2}$$

So we need to correct it by $\chi^1(\frac{x}{\varepsilon})\partial^2 u_1^1/\partial n^2$ with χ^1, η^1 solution of

$$-\nu\Delta_y\chi^1 + \nabla_y\eta^1 = 0, \quad \nabla_y \cdot \chi^1 = 0$$

$$\chi^1|_S = \frac{y_2^2}{2}\vec{e^1}, \quad \chi^1 \ \ y_1 - periodic\ on\ W, \quad lim_{y_2\to+\infty}\chi^1 = C^1$$

Moreover $r^\varepsilon, \rho^\varepsilon$ satisfy the linearized Navier-Stokes equations with a right hand side whose leading term is

$$\varepsilon[\chi_2^0(\frac{\partial u_1^1}{\partial n})^2 + (\frac{x_2}{\varepsilon})(\frac{\partial u_1^1}{\partial n})^2\frac{\partial \chi^0}{\partial y_1}]\vec{e^1}$$

So we need another corrector χ^2 solution of

$$-\nu\Delta_y\chi^2 + \nabla_y\eta^2 = \vec{e^1}[\chi_2^0 + y_2\frac{\partial \chi_2^0}{\partial y_1}], \quad \nabla_y \cdot \chi^2 = 0$$

$$\chi^2|_S = 0, \quad \chi^2 \ \ y_1 - periodic\ on\ W, \quad lim_{y_2\to+\infty}\chi^2 = C^2$$

Finaly to approximate u^ε we are led to introduce u^2 solution of

$$-\nu\varepsilon\Delta u + u.\nabla u + \nabla p = f, \quad \nabla \cdot u = 0$$

$$u.s + \varepsilon C_1^0\frac{\partial u.s}{\partial n} + C_1^2(\frac{u.s}{C_1^0})^2 + C_1^1\frac{\varepsilon}{\nu}\frac{\partial p}{\partial s} = 0|_{\partial\Omega^0}, \quad u.n = 0|_{\partial\Omega^0}.$$

Example
Consider the case where the oscillating boundary does not oscillate but is just a flat plate at a distance ε above the limit flat plate Γ^0. Then the cell width is zero so the 3 cell problems are 1D and they have an analytical solution :

$$\chi^0 = 1, \quad \chi^1 = \frac{1}{2}, \quad \chi^2 = 0, \quad (\eta^2 = y_2)$$

and the constants are $C^0 = 1, C^1 = 0.5, \quad C^2 = 0$. The effective boundary condition on u^2 is

$$u.s + \varepsilon\frac{\partial u.s}{\partial n} + \frac{\varepsilon}{2\nu}\frac{\partial p}{\partial s}$$

which is another form of the Taylor expansion on the boundary mentionned before since $\frac{1}{\nu}\frac{\partial p}{\partial s} = \varepsilon\ \partial^2 u_1^2/\partial n^2$.

4. NUMERICAL TABULATION FOR A WAVY SURFACES

Carrau (1992) and Morrisset (1995) tabulated the cell problem for a compressible flow at high Mach number. We reproduced their simulations at low Mach number with another code. The results are summarized in this table :

Re=10 000	wall	y=0.01	y=0.05	y=0.1
tangent stress	-0.003179	-0.003512	-0.003115	-0.002876
normal stress	7.75155	7.93776	7.93790	7.93794
Re=50 000				
tangent stress	-0.004627	-0.004872	-0.003931	-0.002728
normal stress	7.75261	7.93822	7.93893	7.9390
Re=100 000				
tangent stress	-0.004858	-0.005122	-0.004326	-0.002744
normal stress	7.75147	7.93654	7.93776	7.93802
Re=1000 000				
tangent stress	-0.003845	-0.003876	-0.004079	-0.003008
normal stress	7.75035	7.93501	7.93634	7.93653

Figure 4. *This tabulation of the stress tensor versus the Reynolds number shows also the independence of the mean stress with respect to height*

The geometry and flow visualization are shown on figure 13 and 14 at the end of the paper.

3.1 Test on a flat plate for laminar flow

The previous analysis should work even in the limit of a flat plate whose irregularities tend to zero. Then the periodic cell becomes a vertical line and the computational domain a half plane above the flat plate. So the cell problem is obtained by dropping all tangential derivatives in Navier-Stokes eqs.

$$-\nu\partial_n^2 u + \partial_s p = 0.$$

The solution is a parabolic profile when $\partial_s p$ is constant.

$$u = \frac{\partial_s p}{2\nu}y^2 + \frac{y}{\delta}\left(-\frac{\partial_s p}{2\nu}\delta^2 + u|_{y=\delta}\right).$$

The relation between the normal derivative and itself is easy to find by differentiating the above with repsect to y.

$$\nu\partial_n u + u\frac{\nu}{\delta} + \frac{\partial_s p}{2}\delta = 0.$$

Notice that this boundary condition is the same as the second order condition obtained above. Therefore Domain Decomposition yields a second order condition in this case !

This boundary conditions has been tested for the flow over a flat plate for 2 values of δ, $\delta = 0.01$ or 0.1, with $\nu = 0.003$. ($\delta^+ = \delta\sqrt{\frac{\partial u}{\partial y}\nu^{-1}} = 0.01\sqrt{10^6\ 0.02/9} = \delta \times 0.1414 \times 10^3/3 = 50\delta$)

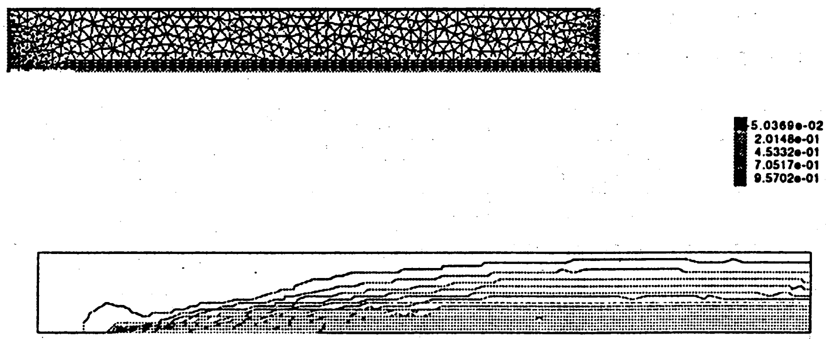

Figure 5 :*Mesh & Navier-Stokes solution with* $u = 0$ *on a flat plate*

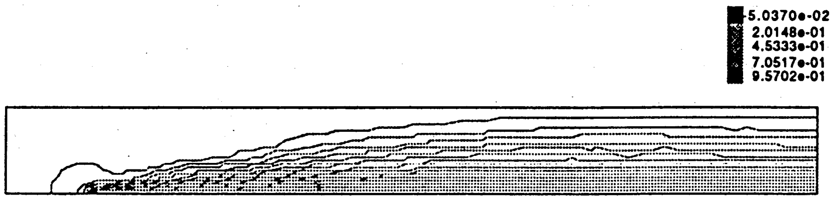

Figure 6 :*Navier-Stokes solution with a laminar wall law and* $\delta = 0.01$

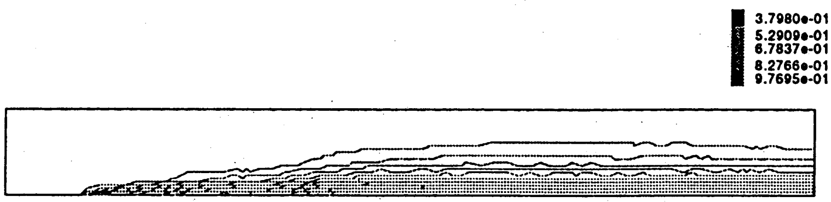

Figure 7 :*Navier-Stokes solution with a laminar wall law and* $\delta = 0.01$

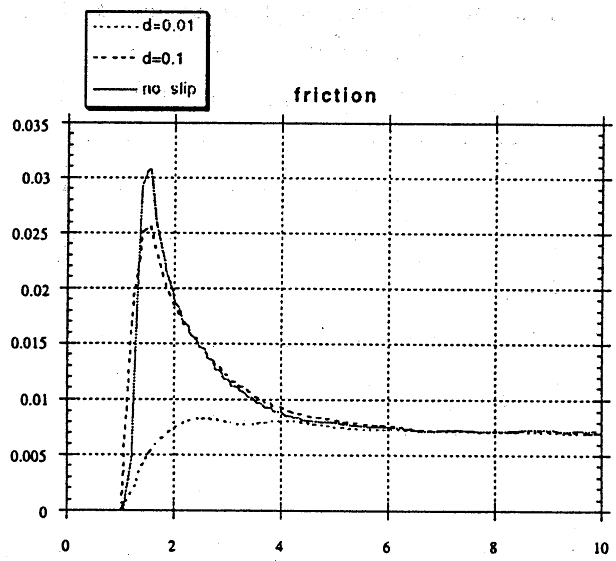

Figure 8 :*Friction at the wall as a function of x*

The numerical results show that it works for a $y^+ < 0.5$ which is much less than the values used for turbulent boundary layers at Re=300 (i.e. $\nu = 1/300$, $h = 1$, $u_\infty = 1$, where h is the height of the computational domain).

This small example also shows the limit of this wall law approach : it is a viscous matching and it has not much to do with Prandtl's boundary layer analysis.

We present also the result of a simulation on a rough flat plate by this method. It amounts to study the dependency of the second order boundary condition with respect to C_1^2.

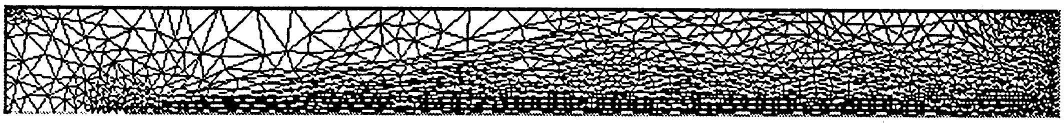

Figure 9 :*Finite Element Mesh adapted for the computation*

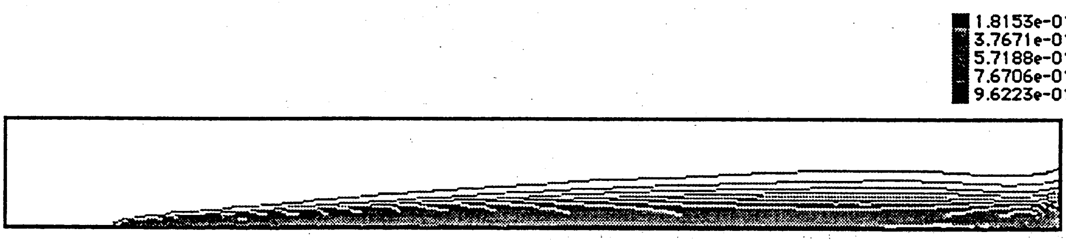

Figure 10 :*Level lines of u_1 for $C^2 = 0.3$*

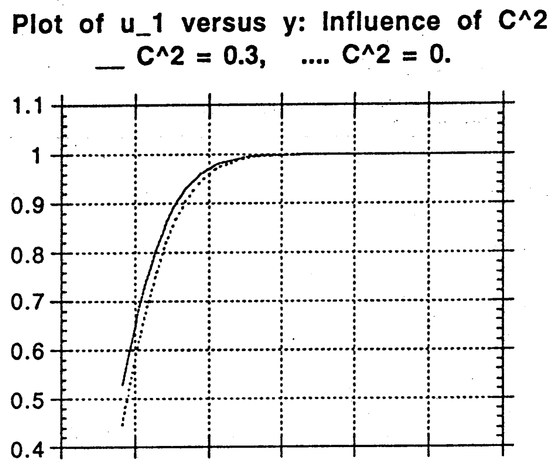

Figure 11 : *Vertical cross section of u_1 for 2 values of C^2*

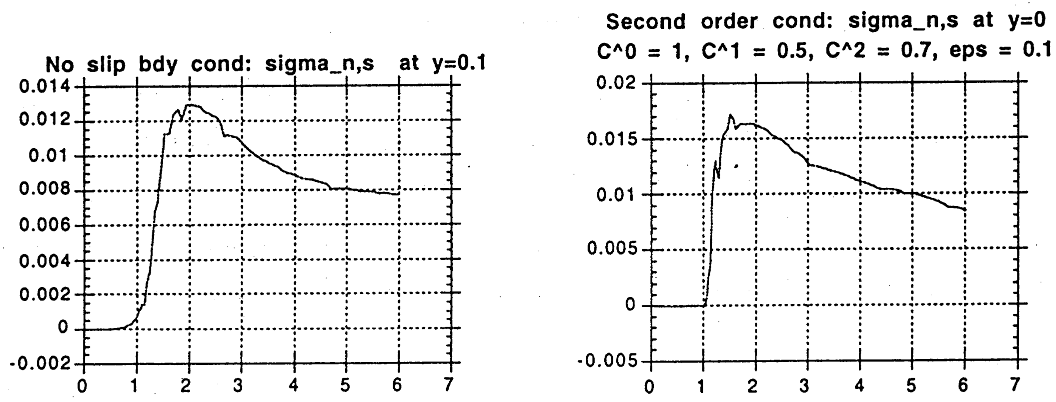

Figure 12 : *Skin friction $\sigma.n.s$ at $y = 0.1$ when the no-slip condition is used (left) and at $y = 0$ when the second order condition is used with eps $= 0.1$*

5. WALL LAWS AND LOW RE CORRECTIONS

5.1 Smooth surface

Let us apply the same idea to the $k - \varepsilon$ model with low Reynolds correction.

$$\mu_T = c_\mu \rho \frac{k^2}{\varepsilon} \quad \text{with} \quad D_t = \partial_t + u\nabla$$

$$E = \frac{1}{2}|\nabla u + \nabla u^T|^2 - \frac{2}{3}|\nabla \cdot u|^2$$

$$D_t k - \frac{\sigma_k}{\rho}\nabla \cdot (\mu_T \nabla k) + k(\frac{\varepsilon}{k} + \frac{2}{3}\nabla \cdot u) = c_\mu \frac{k^2}{\varepsilon} E$$

$$D_t \varepsilon - \frac{\sigma_\varepsilon}{\rho}\nabla \cdot (\mu_T \nabla \varepsilon) + \varepsilon(c_2 \frac{\varepsilon}{k} + \frac{2c_1}{3c_\mu}\nabla \cdot u) = c_1 k E$$

The Low Reynolds number corrections are

$$c'_\mu = f_\mu c_\mu \quad c'_1 = f_1 c_1 \quad c'_2 = f_2 c_2$$

$$f_\mu = (1 - e^{-0.017y\nu^{-1}\sqrt{k}})(1 + 20.5\frac{\nu\varepsilon}{k^2})$$

$$f_1 = 1 + (\frac{0.05}{f_\mu})^3 \quad f_2 = 1 - e^{\nu\varepsilon k^{-2}}$$

We are in fact in the same situation as for the laminar flat plate : two scales, one due to the strong gradients in the normal direction and the other associated with the other gradients. Domain decomposition will give a boundary condition relating the velocity and its gradient on a border at a small distance from the physical boundary.
As in the flat plate case there is no lateral oscillation so the cell problem is on a vertical line, i.e. all tangential derivatives are drop. In the stationary case an analytical solution is found ; it is the wall law when $5 \leq \frac{yu^*}{\nu} \leq 50$:

$$\frac{u}{u^*} = \frac{1}{\chi}\log\frac{yu^*}{\nu} + \beta + y^+\frac{\nu}{\chi u^{*2}}\frac{\partial p}{\partial s}$$

Next eliminate u^*, by differentiating the log law

$$\frac{1}{u^*}\frac{\partial u}{\partial y} = \frac{1}{y\chi} + \frac{1}{\chi u^*}\frac{\partial p}{\partial s}$$

giving

$$u \cdot s = y(\chi\frac{\partial u \cdot s}{\partial n} - \frac{\partial p}{\partial s})(\log(\frac{y^2}{\nu}(\chi\frac{\partial u \cdot s}{\partial n} - \frac{\partial p}{\partial s})) + \beta)$$

which, written at $y = \delta$, gives the required boundary condition.
Usually $\frac{\partial p}{\partial s}$ is dropped because it is small compared with $\frac{\partial u}{\partial n}$, but in Mohammadi-Pironneau(1996) it is shown that this terms helps capture recirculations numerically.

The implementation of this idea has been done using the Reichart law rather than just the log-law because it is valid up to the wall and it is more convenient for recirculation zones where y^+ goes to zero.

$$u^+ = f_{reichart}(y^+) = 2.5log(1 + \kappa y^+) + 7.8(1 - e^{-y^+/11} - \frac{y^+}{11}e^{-0.33y^+}).$$

Our implementation of wall-laws for adiabatic walls, for instance, is in weak form (finite element or finite volume approaches) where the following boundary integrals appear in the momentum and energy equations ($(\vec{s}, \vec{n})$ denotes the local orthogonal basis for a wall node) :

$$\int_{\Gamma_w} (\mathrm{S}.\vec{n})d\sigma,$$

$$\int_{\Gamma_w} (\vec{u}\mathrm{S})\vec{n}d\sigma,$$

where $\mathrm{S} = (\mu + \mu_t)(\nabla u + \nabla u^t - \frac{2}{3}\nabla.uI)$ is the Newtonian strain tensor. We decompose $\mathrm{S}.\vec{n}$ over $(\vec{s}, \vec{n})$:

$$\mathrm{S}.\vec{n} = (\mathrm{S}.\vec{n}.\vec{n})\vec{n} + (\mathrm{S}.\vec{n}.\vec{s}).\vec{t}.$$

In our implementation, the first term (S_{nn}) in the right hand side of these integrals is computed explicitely and the following relations are used :

$$\vec{u}.\vec{n} = 0,$$
$$(\mathrm{S}\vec{n}.\vec{s})\vec{s} = \mu_t \frac{\partial \vec{u}.\vec{s}}{\partial n}\vec{s},$$
$$\vec{u}\mathrm{S}\vec{n} = \mu_t \frac{\partial \vec{u}.\vec{s}}{\partial n}\vec{u}.\vec{s},$$

where $\partial \vec{u}.\vec{s}/\partial n$ is solution of the above relation. This implementation is more suitable for recirculating flows because the direction of the flow is naturally taken into account. For k and ε, we use the following expressions :

$$k = \frac{u_\tau^2}{\sqrt{c_\mu}}\alpha, \quad \varepsilon = \frac{u_\tau^3}{\kappa\delta} min(1, \alpha + \frac{0.2\kappa(1-\alpha)^2}{\sqrt{c_\mu}}),$$

where

$$u_\tau = sqrt\frac{\mu_t}{\rho}|\partial\vec{u}.\vec{s}/\partial n|$$

and $\alpha = min(1, \frac{y^+}{10})$ reproduces the behaviour of k when δ tends to zero. The distance δ is given a priori and is kept constant during the computation. Of course, the pressure correction vanishes with the pressure gradient and we recover the Reichart law.

5.2 Turbulent flow over a wavy surface

When the drag of a wavy surface is assumed proportional to u^2, the domain decompostion approach's answer to the same problem is, as we have seen :

$$-\nu_T \frac{\partial u}{\partial n} = c(\nu_T)u|u|$$

But the wall law being valid at the matching interface this boundary condition could also be used with u given by the law of the wall. It gives

$$u^{*2} = c(\nu_T)u^2 = c(\nu_T)u^{*2}(\frac{1}{\chi}\log\delta^+ + \beta)^2$$

$$\text{i.e.} \quad \beta = c(\nu_T)^{-\frac{1}{2}} - \frac{1}{\chi}\log\delta^+$$

So the effect of the roughness is to change the value of $c(\nu_T)$ hence to shift the value of β by $c_{wavy}(\nu_T)^{-\frac{1}{2}} - c_{flat}(\nu_T)^{-\frac{1}{2}}$

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APPENDIX : Stokes Flow by domain decomposition

For Stokes flow the mean flow away from the rough surface is found by

$$- \nu\Delta u^0 + \nabla p^0 = 0, \quad \nabla\cdot u^0 = 0 \ \text{ in } \ \Omega,$$
$$- \nu\langle\chi\rangle\partial_n u^0 + p^0 n + \frac{1}{\varepsilon}u^0 = 0 \ \text{ on } \ \Sigma, \quad u^0|_{\Gamma_1} = g$$

where the matrix $\chi = \{\chi^1, \chi^2, \chi^3\}$ has χ^i solution of

$$-\nu\Delta\chi + \nabla\eta = 0, \quad \nabla.\chi = 0,$$

with periodic conditions on the lateral boundaries, $\chi = 0$ on the lower boundary, and on the upper boundary S of the cell domain

$$-\nu\partial_n\chi^i + \eta^i n = E^i, \ \text{ with } \ E^i_j = \delta_{ij}$$

Because $\nabla.\chi = 0$ we have that $\langle\chi.n\rangle|_S = 0$ so that $u^0.n = 0$ on Σ.
The main result (Achdou-Pironneau (1994)) compares the exact solution u^ε with the solution u^0 above a mean surface Σ with a Frechet boundary condition

$$||u^\varepsilon - u^0||_{\Omega^0} \leq C(\varepsilon\|\partial_s\chi\|_{0,S} + \varepsilon^{3/2})$$

where χ is the solution of the cell problem which defines the constant in the Frechet boundary condition. This result shows that the smooth artificial boundary Σ should be sufficiently far from the wavy boundary so as to have $\|\partial_s\chi\|_{0,S} = 0(\epsilon^{\frac{1}{2}})$ which is possible because χ tends to a function independent of s at infinity.

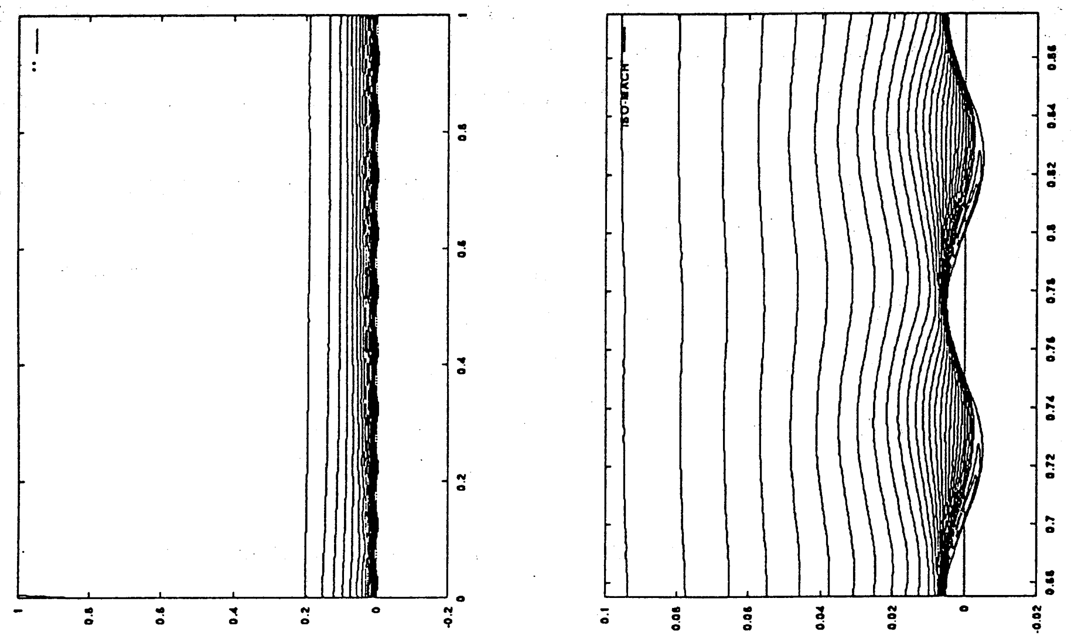

Figure 13. *Mach lines and zoom of the center part for the flow over a rough boundary which was used to compute the table of figure 4*

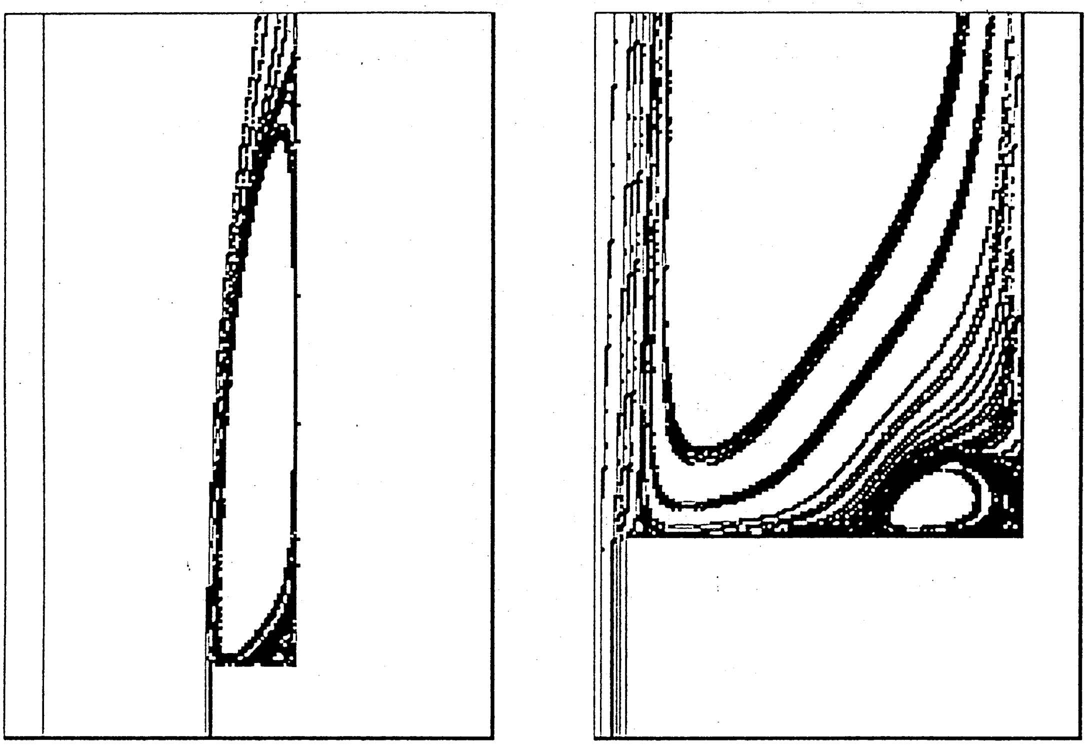

Figure 14. *Mach lines and zoom of the recirculating part for the flow over a backward step with k-epsilon modelling and wall laws with a pressure term. The second vortex is captured with the wall law which includes a pressure gradient.*